The volume of the cone v—DB is

$$V_0 = \pi \frac{R+r}{2}\cdot\frac{\sin\varphi}{\sin\theta}\left(R\,r\right)^{\frac{1}{2}}\frac{h\sin\theta}{3} = \tfrac{1}{3}\pi h\sin\varphi\,\frac{R+r}{2}\left(R\,r\right)^{\frac{1}{2}},$$

but $$\frac{R+r}{2}=\frac{R\,r}{x};\text{ hence } V_0 = \frac{1}{3x}\,\pi h\sin\varphi\,R^{\frac{3}{2}}r^{\frac{3}{2}}.$$

Now $$vs = \frac{hr}{x}\text{ and } vk = \frac{h\,R}{x},$$ hence

$$\text{cone } v—AB = V_1 = \frac{1}{3x}\,\pi h\sin\varphi\,r^3,\ \text{cone } v—CD = V_2 = \frac{1}{3x}\,\pi h\sin\varphi\,R_{\mathbf{3}}.$$

$$\text{Vol. } ABD = V_0 - V_1 = \frac{1}{3x}\,\pi h\sin\varphi\left(R^{\frac{3}{2}} - r^{\frac{3}{2}}\right)r^{\frac{3}{2}},$$

$$\text{Vol. } BDC = V_2 - V_0 = \frac{1}{3x}\,\pi h\sin\varphi\left(R^{\frac{3}{2}} - r^{\frac{3}{2}}\right)R^{\frac{3}{2}},$$

hence $$\frac{\text{Vol. } ABD}{\text{Vol. } BDC} = \frac{\sqrt{r^3}}{\sqrt{R^3}}.$$

From the volumes of the three cones we see that the elliptical cone is a mean proportional between the other two. (This striking analogy between the cones $v — AB$, $v — BD$, $v — CD$, and the triangles AvB, DvB, DvC, I had never before noticed, though it must have been well known. Neither had I ever observed that the semi-conjugute axis of *such* a conic section is a mean proportional between the radii of the bases of the frustrum.)

It is obvious that the plane AC divides the frustrum in the same ratio as the plane BD.

If the altitude of the frustrum is p we have $h\sin\varphi \div x = p \div (R—r)$ and

$$V_0 = \tfrac{1}{3}\pi p[\sqrt{(R^3r^3)} \div (R—r)],$$

$$V_1 = \tfrac{1}{3}\pi p[r^3 \div (R—r)],\ V_2 = \tfrac{1}{3}\pi p[R^3 \div (R—r)],$$

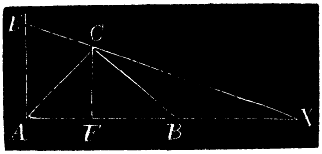
$$V_0 — V_1 = \tfrac{1}{3}\pi p\,\frac{\sqrt{R^3} — \sqrt{r^3}}{R—r}\,r^{\frac{3}{2}},$$

$$V_2 — V_0 = \tfrac{1}{3}\pi p\,\frac{\sqrt{R^3} — \sqrt{r^3}}{R—r}\,R^{\frac{3}{2}}.$$

SOLUTION OF PROBLEM 255.

BY PROF. W. W. HENDRICKSON, NAVAL ACADEMY, ANNAPOLIS, MD.

DENOTING the distance AB by a, and taking the axes as represented in the figure, the equations to the lines AC and BC are (1.) $y = x\tan(\varphi + a)$,



(2.) $y = \tan 2\varphi(a - x)$. Eliminating φ, the equation to the locus is $3x^2y - y^3 - 2axy = \tan 2\alpha(x^3 - 3xy^2 - ax^2 + ay^2)$.

By the conditions of the problem there will be an asymptote whenever AC and BC become parallel; that is, when $\alpha + \varphi = n\pi - 2\varphi$; from this $\varphi = \frac{1}{3}(n\pi - \alpha)$, $\alpha + \varphi = \frac{1}{3}(n\pi + 2\alpha)$, and the direction ratios of the asymptotes are the tangents of the angles $\frac{2}{3}\alpha$, $60^\circ + \frac{2}{3}\alpha$, and $120^\circ + \frac{2}{3}\alpha$. The asymptotes all pass through the point $(\frac{1}{3}a, 0)$ as will be seen by moving the origin to that point, when all the terms of the second degree in the equation to the locus will disappear. The equation to an asymptote is $y = \tan \frac{1}{3}(n\pi + 2\alpha)(x - \frac{1}{3}a)$.

In equations (1.) and (2.) let $\alpha = 45^\circ$, and denote $\tan \varphi$ by m, then the co-ordinates of C are

$$x_1 = \frac{2am}{m^2+4m+1},\ y_1 = \frac{2am(1+m)}{(m^2+4m+1)(1-m)};\ \tan(90^\circ+4\varphi) = \frac{1-6m^2+m^4}{4m(m^2-1)},$$

whence (after reduction) the equation to CE is

$$4m(1-m^2)y + (m^4 - 6m^2 + 1)x = 2am(m^2+1). \tag{3}$$

Differentiating in reference to m,

$$(2-6m^2)y + (2m^3 - 6m)x = a(3m^2+1). \tag{4}$$

From (3) and (4), $x = \dfrac{-8am^3}{(m^2+1)^3}$, $y = \dfrac{-a(m^6+9m^4-9m^2-1)}{2(m^2+1)^3}$, squaring and adding, $x^2+y^2 = \dfrac{a^2}{4} \cdot \dfrac{m^4+14m^2+1}{m^4+2m^2+1}$, whence $\dfrac{m^2}{(m^2+1)^2} = \dfrac{4x^2+4y^2-a^2}{12a^2}$; the value of $m \div (m^2+1)$ obtained from this and substituted in the value of x, gives

$$(4x^2+4xy^2-a^2)^3 = 27a^4x^2,$$

which is the eq'n to the envelop of (3) and is the two-cusped epicycloid.

[Prof. Johnson writes, in relation to the solution of this problem, "The envelop is really the two-cusped epicycloid. Also, the result cannot be independent of α, since for any other value than 45° the line EC recedes to infinity when C recedes to infinity on one of the asymptotes of the cubic locus, and this indicates parabolic branches of the envelop." Prof. Johnson proposes the following:

Problem.—"Supposing the fig. on p. 91 constructed for $\alpha = 45^\circ$ as in the second part of Prob. 255, let the circle whose centre is A and radius AB intersect EC in M and N. Prove that as φ varies M and N move with uniform but unequal rates on the circumference of the circle.

"*Note.*—This theorem reduces the quest. in Prob. 255 to the determination of the envelop of a chord whose extremities move uniformly in a circle.

"This envelop is always an epicycloid."]